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Dynamics of relaxing systems subjected to nonlinear interactions

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The combination of the Fermi map system and half a stadium is studied to determine the effect of additional nonlinearity in the well known Fermi acceleration problem. The relaxation in the Fermi-stadium map with different R's is compared to that in the Fermi map. The relaxation is found retarded for different values of R. After a crossover time, the Fermi relaxation can be approximated by an exponential function, while the Fermi-stadium relaxation can be approximated by a stretched-exponential function. The fractional exponent β decreases further from unity with increasing nonlinearity. The result bears strong similarity to the basic features suggested by the coupling model and seen experimentally in glass-forming materials by neutron scattering.

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INTRODUCTION

Recent results [1] have shown that studies of nonlinear dynamical systems can enhance the understanding of some fundamental problems in physics, such as stability of the solar system, phase transitions, turbulence, and the ergodic problems in statistical mechanics. One common characteristic of such systems is the irreversibility of the dynamics due to sensitive dependence of initial conditions. Simplified models for systems such as the Fermi acceleration [2] have indicated that such systems relax to equilibrium through evolutions as a Markov process. The evolution can be determined from a Fokker-Planck equation. It would be of interest to study the relaxation of a more complex system, consisting of a number of such systems coupled together by nonlinear interactions. Such a study may be beneficial to the understanding of relaxation processes in glass-forming viscous liquids, polymers, and ionic conductors, to name a few. These problems in condensed matter physics, physical chemical, and materials science involve irreversible processes in densely packed interacting systems [3]. The interactions in these systems come from nonlinear potentials such as that of Lennard-Jones in liquids and polymers and that of Coulomb in vitreous fast ionic conductors.

Recently we analyzed a system of interacting arrays of globally coupled oscillators [4]. The relaxation in the system of interacting arrays was compared to that in an array not subject to interaction with others. The relaxation of the latter was found to be exponential in time, while that of the former was slowed down and its time dependence departed from being exponential. There exists a crossover time t_c before which the relaxation of the interacting arrays remains exponential. However, beyond t_c the relaxation is no longer exponential but well approximated by a stretched exponential $\exp[-(t/\tau)^{\beta}]$. The fractional exponent β was found to decrease further from unit with increasing interaction strength. The result bears strong similarity to the basic features suggested by the coupling model of relaxation in glass-forming materials [5–7] and seen experimentally by neutron scattering [8] for relaxation in densely packed interacting molecules in glass-forming liquids. The possibility that the results of the coupling model originate from nonlinear Hamiltonian dynamics has existed ever since the first proposal [5] that makes connection to quantum chaology [9]. Later relations to nonlinear classical mechanics were made [4,6,7]. Recently, Tsironis and Aubry [10] have studied relaxation properties of one-dimensional nonlinear lattices and found also nonexponential lattice energy relaxation.

The interacting arrays model [4] provides a useful step in applying nonlinear dynamical models to study the irreversible processes of real physical systems. However, a nonlinearly coupled many body system would be very complicated to study. Introducing more nonlinearities to the simpler system would serve the purpose of providing insights into what would happen in the complex problem. It is therefore a well-posed problem to study the effect of additional nonlinearity in well known problems such as the Fermi acceleration. What we can learn from the solution of the problem should shed some light on the nonlinearly coupled many body problem.

FERMI MAP

A two-dimensional map can be used to model a realistic problem such as the cosmic ray acceleration mechanism proposed by Fermi [2], in which charged particles are accelerated by collisions with moving magnetic field structures. In the model, shown in Fig. 1, a ball bounces back and forth between a fixed (dissipative) wall and an (elastic) wall oscillating sinusoidally with $x=a\cos\omega t$. The motion of the ball can be described by the Fermi map [11–15]. The map can be written as

$$u_{n+1}=|(1-\delta)u_n+\sin\varphi_n|,$$

$$\varphi_{n+1}=\varphi_n+2\pi M/u_{n+1},$$

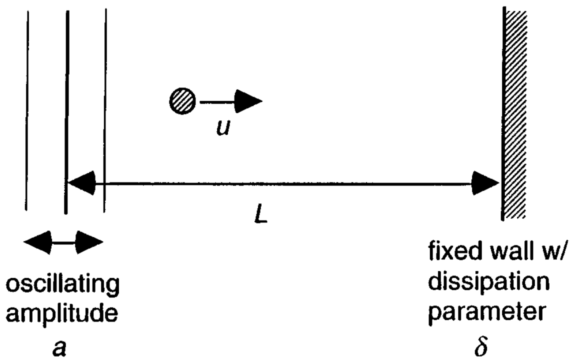

FIG. 1. One-dimensional dissipative Fermi map.

where u_n is the normalized ball velocity and φ_n is the phase of the oscillating wall just before the nth collision of the ball with the oscillating wall, and $M=L/16a$ is the normalized distance between the two walls. Tsang and Lieberman [12,13], considering the case where $M \gg 1$ and $\delta \ll 1$, calculated that almost all initial distributions in velocities $f_0(u)$ evolve to the invariant one $f(u) \sim \exp(-2\delta u^2)$. Numerical computations for various values of M were reported to be in agreement with such calculations. Assuming that the phase-averaged distribution evolves with time (iteration number) n as a Markov process in u alone, the evolution can be written in the form of a Fokker-Planck equation,

$$\frac{\partial f}{\partial n} = -\frac{\partial}{\partial u} Bf + \frac{1}{2}\frac{\partial^2}{\partial u^2} Df,$$

where the friction coefficient is

$$B(u) = \frac{1}{2\pi}\int_0^{2\pi} \Delta u \, d\varphi = -\delta u,$$

and the diffusion coefficient is

$$D(u) = \frac{1}{2\pi}\int_0^{2\pi} (\Delta u)^2 d\varphi = \frac{1}{2} + \delta^2 u^2.$$

Consider an ensemble of particles with an initial velocity distribution almost being a delta function $\delta(u)$. As time evolves, the narrow Gaussian distribution widens, approaching the invariant Gaussian distribution. The average energy relaxes to the equilibrium energy *exponentially* in time.

FERMI-STADIUM MAP

To study one more dimension to the Fermi map, we add two more elastic walls to the existing ones. As shown in Fig. 2, the width between the new walls is 2, which is much smaller than L. Although the motion of the ball is now two dimensional, the Fermi system with the two walls added is just as simple as the one-dimensional Fermi map (since ν remains constant forever) unless we add more nonlinearity to the system by, say, introducing some curvature at the two ends of the fixed dissipative wall. In particular, the right-angled corners are replaced by quadrants of a circle with radius R [see Fig. 2(b)], which is much smaller than 1. The composite system is actually the combination of the Fermi map system and half a stadium. Both the Fermi map and the

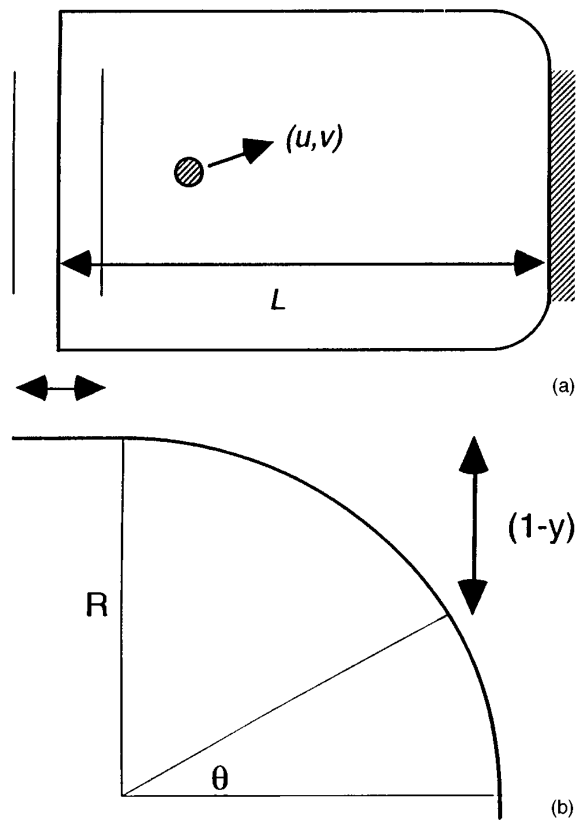

FIG. 2. (a) Two-dimensional Fermi-stadium map; (b) a curved corner of the added stadium portion.

stadium problems [16–20] are classical problems known to many in nonlinear dynamics, and we call this map for the system the Fermi-stadium map. The dynamics of the map is no different from the Fermi map except when the ball collides with the curved corners. The resulting orbits are deflected by twice the angle between the tangent there and the dissipative wall. Before considering the effect of the deflection at the curved corners, the map can be written as

$$u' = (1-\delta)u_n + \sin\varphi_n,$$

$$\varphi' = \varphi_n + 2\pi M/u',$$

$$\nu' = \nu,$$

$$y' = y + \nu L/u \quad (\text{mod}4)$$

.

The modulo 4 restricts y between -2 and 2, in light of the fact that one collision at the ceiling and another at the floor are equivalent to a translation of 4 in the coordinate y. For an extra collision, the value of y would be larger than 1 or smaller than -1. To adjust y again so that it is between -1 and 1, we do the following:

$$\text{if} \quad y' > 1 \quad \text{then} \quad y'' = 2 - y,$$

$$\text{if} \quad y' < -1 \quad \text{then} \quad y'' = -2 - y.$$

We then add the effect due to collisions at the curved corners. If

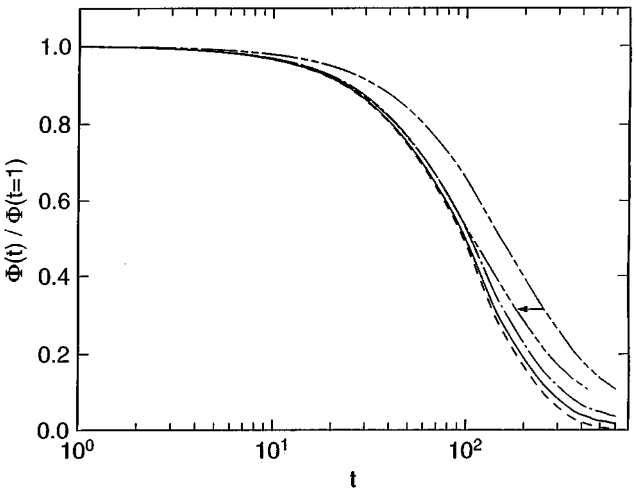

FIG. 3. Numerical result of the Fermi-stadium map with parameters $\delta=0.01$, L, and $M=100$, for various values of R (0, 0.02, 0.1, and 0.5, from left to right). The normalized difference Φ between the mean energy and the equilibrium energy is plotted against time t. The arrow indicates a rescaling of the $R=0.5$ curve to show a longer coincidence with the Fermi curve.

$$y''\left\{\begin{matrix} >1-R \\ <-(1-R) \end{matrix}\right\},$$

then

$$\begin{pmatrix} u'' \\ v'' \end{pmatrix}=\begin{pmatrix} \cos 2\theta & \mp\sin 2\theta \\ \pm\sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix},$$

where $\theta=\sin^{-1}[(|y''|-1+R)/R]$. The new values at the $(n+1)$st collision with the oscillating wall are then

$$u_{n+1}=|u''|,$$

$$v_{n+1}=v'',$$

$$y_{n+1}=|y''|,$$

$$\varphi_{n+1}=\varphi'.$$

RELAXATION OF AVERAGE ENERGY

To study the relaxation of the average energy in the Fermi-stadium map, we observed the time evolution of 10 000 initial conditions. The initial distribution $f_0(u)$ is such that $u_0 \ll 1$ and $v_0 \ll u_0$. In order to smooth the high-frequency fluctuations in time, we use a running average over 50 time steps. We then compare the relaxation in the Fermi-stadium map with different R's to that in the Fermi map.

NUMERICAL RESULTS

Figure 3 shows the numerical result of the Fermi-stadium map with parameters $\delta=0.01$, L and $M=100$, for various values of R (0, 0.02, 0.1, and 0.5). The normalized difference Φ between the mean energy and the equilibrium energy is plotted against time t. The relaxation of Φ is shown to be slowed down when R is nonzero and the degree of slowing down increases with R. Note that for $R=0$ (the dashed line at the bottom), the map is reduced to the Fermi map and there is no slowing down of the relaxation. At $t=140$, the calculated solid curve, corresponding to $R=0.02$, begins to significantly deviate from the Fermi curve. The next curve above corresponds to $R=0.1$, and it begins to deviate significantly from the Fermi curve starting at $t=120$. For $R=0.5$, since the change from the Fermi map is substantial, we rescale the curve horizontally so that it coincides for a significant portion with the Fermi curve. The rescaling is shown by the arrow in Fig. 3. At $t=85$, the curve corresponding to $R=0.5$ begins to deviate significantly from the Fermi curve. Thus, there exists a crossover time t_c (the magnitude of which depends on R), before which the Fermi-stadium curve decays exponentially like the Fermi curve, but thereafter it deviates from the Fermi curve. The existence of such a crossover time from simple to coupled dynamics is the characteristic of the coupling model [5–7] and was found also in the relaxation of the interacting arrays [4].

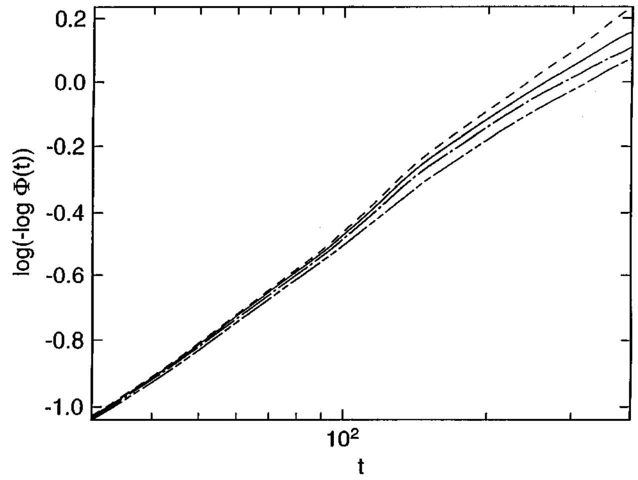

FIG. 4. $\log_{10}(-\log_{10}\Phi)$ against $\log_{10}t$. The curves become nearly straight after a crossover time t_c, which depends on R, and can be fitted to straight lines of slopes $\beta=1.0$, 0.86, 0.8, and 0.78, respectively, for $R=0$, 0.02, 0.1, and 0.5 (from left to right).

To study the nature of the relaxation, we plot $\log_{10}(-\log_{10}\Phi)$ against $\log_{10}t$ in Fig. 4. After a crossover time t_c (which depends on R), the curves become nearly straight. The portion of the curves can be fitted to straight lines of slopes $\beta=1.0$, 0.86, 0.8, and 0.78, respectively, for $R=0$, 0.02, 0.1, and 0.5. Figure 5 shows that the portion of curves corresponding to large t can be approximated by simple functions. In particular, the Fermi curve (solid curve below) is shown approximated by the (dotted) exponential function. The Fermi-stadium curve, on the other hand, is shown approximated by a (dotted) stretched-exponential function. The dashed curve is an exponential indicating a failure to fit the Fermi-stadium curve.

DISCUSSION AND CONCLUSION

The effect of additional nonlinearity, in the form of half a stadium, to the Fermi map, has been studied. For $R=0$, the problem reduces to the Fermi map, which relaxes exponentially [1,11,12], i.e., for large t, $\Phi\sim\exp(-t/\tau)$. For nonzero R's, it is found that $\Phi\sim\exp[-(t/\tau')^{\beta}]$ for $t>t_c$. Thus the

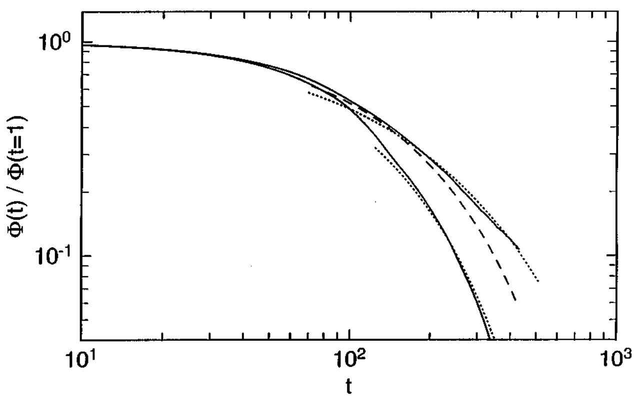

FIG. 5. The Fermi curve (solid curve below) approximated by the (dotted) exponential function, and the Fermi-stadium curve, with $R=0.5$, approximated by a (dotted) stretched-exponential function. The dashed curve is an exponential indicating a failure to fit the Fermi-stadium curve.

relaxation of the Fermi-stadium map proceeds with a stretched exponential time dependence at long times starting at $t \approx t_c$. The nonlinearity from the stadium is observed to slow down the relaxation of the original system. As expected, the fractional exponent decreases further from unity with increasing nonlinearity or larger R's.

In the study of relaxation in real systems in physics, chemistry, and materials science, it is found that a system without interactions usually relaxes exponentially. However, when such systems are densely packed and interacting with each other the relaxation proceeds differently and exhibits many fascinating properties. A coupling model has been proposed to explain these properties [7,21]. This model is based on the hypothesis that an interacting system relaxes initially exponentially until at time t_c, but stretched exponentially afterwards with continuity of the correlation function at the time of crossover [5–7]. A recent neutron scattering measurement on a polymer has shown direct experimental evidence for this hypothesis [8]. In our present work, the addition of half of a stadium to the Fermi model to make it two dimensional introduces additional nonlinearity that has similar effects on the relaxation towards equilibrium as many body interactions have on relaxation in densely packed systems. Since the numerical result obtained bears strong similarity to the basic features suggested by the coupling model [5–7] and seen in neutron scattering experiment [8], the Fermi-stadium map provides a useful first step in applying nonlinear dynamical models to the study of irreversible processes of real physical systems in physics, chemistry, and materials science.

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